A Proof That Zeilberger Missed: A New Proof Of An Identity By Chaundy And Bullard Based On The Wilf-Zeilberger Method

YiJun Chen

Abstract

In this paper, based on the Wilf-Zeilberger theory, a very succinct new proof of an identity by Chaundy and Bullard is given.

In their paper [4], T. H. Koornwinder and M. J. Schlosser have given detailed explanations of the historical developments, different proofs (in fact, they provide seven different proofs), and different generalizations of the identity

$$1 = (1-x)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}x^k + x^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-x)^k, \tag{1}$$

which is attributed to Chaundy and Bullard. In particular, we can learn in [4] that the special case where $m = n$ plays a key role in Daubechies's theory of wavelets of compact support(see [2]or [3]). A very succinct probabilistic proof of this special case was given by Zeilberger (see [6]). As pointed out in [4], the same argument in [6] can also be used when $m \neq n$.

In this paper, I will give a new proof of the identity (1) based on the Wilf-Zeilberger method (or WZ method for short, see [5]). This is a very succinct beautiful proof which Zeilberger missed. I will use the following proposition (see [1]) to prove identity (1). Given a WZ pair $(F(n,k), G(n,k))$, that is

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k), \tag{2}$$

then for all $m, n \in N_0 = N \cup \{0\}$, we have

$$\sum_{k=0}^{m} F(n,k) = \sum_{j=0}^{n-1}[G(j,m+1) - G(j,0)] + \sum_{k=0}^{m} F(0,k). \tag{3}$$

Now set

$$F(n,k) = \binom{n+k}{k}x^k(1-x)^{n+1}$$

and

$$R(n,k) = -\frac{k}{n+1}, G(n,k) = R(n,k)F(n,k).$$

It is easy to verify that $(F(n,k), G(n,k))$ is a WZ pair and that the following statements are true.

(a) For all $j \in N_0$, $G(j,0) = 0$.

(b) For all $m \in N_0$, $\sum\limits_{k=0}^{m} F(0,k) = 1 - x^{m+1}$.

By (3), (a), and (b), we have

$$\begin{aligned}
\sum_{k=0}^{m} F(n,k) &= -x^{m+1}\sum_{j=1}^{n}\binom{m+j}{j}(1-x)^j + 1 - x^{m+1} \\
&= -x^{m+1}\sum_{j=0}^{n}\binom{m+j}{j}(1-x)^j + 1.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&(1-x)^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}x^k + x^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-x)^k \\
=\ & -x^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-x)^k + 1 + x^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(1-x)^k \\
=\ & 1,
\end{aligned}$$

and the proof is complete.

References

[1] Y. J. Chen, WZ method and a formula for the partial sum of a binomial series(to appear).

[2] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988) 909–996.

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&= -x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1-x)^k + 1 + x^{m+1} \sum_{k=0}^{n} \binom{m+k}{k} (1-x)^k \\
&= 1,
\end{aligned}$$

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